

FÖLNER SEQUENCES AND SUM-FREE SETS

SEAN EBERHARD

ABSTRACT. Erdős showed that every set of n positive integers contains a subset of size at least $n/(k+1)$ containing no solutions to $x_1 + \dots + x_k = y$. We prove that the constant $1/(k+1)$ here is best possible by showing that if (F_m) is a multiplicative Følner sequence in \mathbf{N} then F_m has no k -sum-free subset of size greater than $(1/(k+1) + o(1))|F_m|$.

1. INTRODUCTION

Let $k \geq 2$ be an integer. A subset A of an abelian group is called k -sum-free if there do not exist $a_1, \dots, a_k \in A$ such that $a_1 + \dots + a_k \in A$. In 1965 Erdős [Erd65] proved with the following ingenious argument that every set A of n positive integers has a k -sum-free subset of size at least $n/(k+1)$. Since the open interval $S \subset \mathbf{R}/\mathbf{Z}$ of length $1/(k+1)$ centred at $1/(2k-2)$ is k -sum-free, it follows that for each $x \in \mathbf{R}/\mathbf{Z}$ the set A_x of $a \in A$ such that $ax \in S$ is also k -sum-free. But if x is chosen uniformly at random from \mathbf{R}/\mathbf{Z} then for each $a \in A$ the product ax also has the uniform distribution, so by linearity of expectation the expected size of A_x is $n/(k+1)$. In particular $|A_x| \geq n/(k+1)$ for some x .

Our main theorem is that the constant $1/(k+1)$ in this theorem cannot be improved: for every $\varepsilon > 0$ there is a set of n positive integers containing no k -sum-free subset of size greater than $(1/(k+1) + \varepsilon)n$. In fact, we can give explicit examples of sets with no large k -sum-free subsets. Call a sequence (F_m) of subsets $F_m \subset \mathbf{N}$ a *Følner sequence* in (\mathbf{N}, \cdot) if, for every fixed $a \in \mathbf{N}$,

$$\frac{|(a \cdot F_m) \Delta F_m|}{|F_m|} \longrightarrow 0 \quad \text{as } m \rightarrow \infty,$$

where Δ denotes symmetric difference. For example, if

$$F_m = \{p_1^{e_1} \cdots p_m^{e_m} : 0 \leq e_i < m\},$$

where p_1, p_2, \dots are the primes, then (F_m) is a Følner sequence in (\mathbf{N}, \cdot) . For any such sequence, the sets F_m eventually have no large k -sum-free subsets.

Theorem 1. *If (F_m) is a Følner sequence in (\mathbf{N}, \cdot) then F_m has no k -sum-free subset of size greater than $(1/(k+1) + o(1))|F_m|$.*

We begin by giving a quick deduction of the case $k = 2$ of this theorem from previous work. We then give two proofs in the general case: one short, infinitary, and ineffective, the other longer, finitary, and effective, though with poor bounds. In both proofs we rely on the theorem of Łuczak and Schoen [LS97] that every maximal k -sum-free subset of \mathbf{N} of upper density greater than $1/(k+1)$ is periodic. In the first proof we use this theorem as a black box, while for the second we prove a finitary version by closely following [LS97].

Ben Green, Freddie Manners, and I [EGM13] recently proved that there is a set of n positive integers containing no 2-sum-free subset of size larger than $(1/3 + o(1))n$. The method we used extends to the case of 3-sum-free sets with a little work, but the method does not seem to extend easily to k -sum-free sets for $k > 3$. Until now, the best result known in the case $k > 3$ was the theorem of Bourgain [Bou97] that if δ_k is the largest constant such that every set of n positive integers contains a k -sum-free set of size at least $\delta_k n$ then $\delta_k \rightarrow 0$ as $k \rightarrow \infty$.

2. DEDUCTION OF THE CASE $k = 2$ FROM PREVIOUS WORK

Fix a Følner sequence (F_m) in (\mathbf{N}, \cdot) . The Følner property will be used in the following way: if $x \in F_m$ is uniformly random and $E(x)$ is some event depending on x , then for every fixed $a \in \mathbf{N}$ we have

$$|\mathbf{P}(E(ax)) - \mathbf{P}(E(x))| \leq \frac{|(a \cdot F_m) \triangle F_m|}{|F_m|} \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

This allows us to imitate Erdős's argument with (F_m) in place of \mathbf{R}/\mathbf{Z} , as in the following lemma.

Lemma 2.1. *The following two statements are equivalent.*

- (1) *For infinitely many m , F_m has a k -sum-free subset of size at least $\delta|F_m|$.*
- (2) *For every finite $A \subset \mathbf{N}$, A has a k -sum-free subset of size at least $\delta|A|$.*

Proof. Of course (2) trivially implies (1), so it suffices to prove (1) implies (2). Suppose F_m has a k -sum-free subset S_m of size at least $\delta|F_m|$. For $x \in F_m$, let A_x be the set of all $a \in A$ such that $ax \in S_m$. Then A_x is k -sum-free, and if we choose $x \in F_m$ uniformly at random then the expected size of A_x is

$$\begin{aligned} \mathbf{E}(|A_x|) &= \sum_{a \in A} \mathbf{P}(a \in A_x) = \sum_{a \in A} \mathbf{P}(ax \in S_m) \\ &\geq \sum_{a \in A} \left(\mathbf{P}(x \in S_m) - \frac{|(a \cdot F_m) \triangle F_m|}{|F_m|} \right) \\ &\geq \delta|A| - \sum_{a \in A} \frac{|(a \cdot F_m) \triangle F_m|}{|F_m|}. \end{aligned}$$

Hence from the Følner property and the integrality of $|A_x|$ it follows that for sufficiently large m there is some $x \in F_m$ such that $|A_x| \geq \delta|A|$. \square

By [EGM13], for every $\varepsilon > 0$ there is a set A of n positive integers with no 2-sum-free subset of size greater than $(1/3 + \varepsilon)n$. From this and the above lemma we deduce the case $k = 2$ of Theorem 1.

3. INFINITARY PROOF OF THEOREM 1

In this section we assume basic familiarity with ultrafilters, and in particular Loeb measure. The reader needing an introduction might refer to Bergelson and Tao [BT13, Section 2].

Again fix a Følner sequence (F_m) in (\mathbf{N}, \cdot) , and assume for infinitely many m that F_m has a k -sum-free subset S_m of size at least $\delta|F_m|$. By passing to a subsequence we may assume this holds for all m .

Lemma 3.1. *There is an abelian group X , a σ -algebra Σ of subsets of X , a probability measure μ on Σ , and a set $S \in \Sigma$ such that (1) for every $a \in \mathbf{N}$ the map $x \mapsto ax$ is Σ -measurable and μ -preserving, and (2) S is k -sum-free and $\mu(S) \geq \delta$.*

Proof. Let $p \in \beta\mathbf{N} \setminus \mathbf{N}$ be a nonprincipal ultrafilter, let X be the ultraproduct $\prod_{m \rightarrow p} \mathbf{Z}$, and let Σ be the Loeb σ -algebra on X . Defining μ_m on subsets of \mathbf{Z} by

$$\mu_m(A) = |A \cap F_m|/|F_m|,$$

let μ be the Loeb measure induced by (μ_m) . Let S be the internal set $\prod_{m \rightarrow p} S_m$.

To verify (1), note that $x \mapsto ax$ sends internal sets to internal sets, so it is measurable. Moreover $x \mapsto ax$ approximately preserves μ_m by the Følner property, so it exactly preserves the Loeb measure μ . For (2), it follows from the basic properties of ultrafilters that S is k -sum-free, and by definition of μ we have

$$\mu(S) = \text{st} \left(\lim_{m \rightarrow p} \mu_m(S_m) \right) \geq \delta. \quad \square$$

We define the *upper density* of a set $A \subset \mathbf{N}$ by

$$\overline{d}(A) = \limsup_{n \rightarrow \infty} \frac{|A \cap \{1, \dots, n\}|}{n},$$

and we define its *upper density on multiples* by

$$\tilde{d}(A) = \limsup_{N \rightarrow \infty} \overline{d}(A/N!),$$

where $A/N! = \{a \in \mathbf{N} : N!a \in A\}$. Equivalently,

$$\tilde{d}(A) = \limsup_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{|A \cap \{N!, 2N!, \dots, nN!\}|}{n}.$$

Lemma 3.2. *There is a k -sum-free subset A of \mathbf{N} such that $\tilde{d}(A) \geq \delta$.*

Proof. Let X , Σ , μ , and S be as in the previous lemma. For $x \in X$, let A_x be the set of all $a \in \mathbf{N}$ such that $ax \in S$. Then A_x is k -sum-free, $\tilde{d}(A_x)$ is a Σ -measurable function of x , and if x is chosen randomly with law μ then, by Fatou's lemma,

$$\begin{aligned}
\mathbf{E}(\tilde{d}(A_x)) &\geq \limsup_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{E} \left(\frac{|A_x \cap \{N!, 2N!, \dots, nN!\}|}{n} \right) \\
&= \limsup_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{a=1}^n \mathbf{P}(aN! \in A_x) \\
&= \limsup_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{a=1}^n \mathbf{P}(aN!x \in S) \\
&= \limsup_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{a=1}^n \mathbf{P}(x \in S) \\
&= \mu(S) \geq \delta. \quad \square
\end{aligned}$$

To complete the proof of Theorem 1, it suffices to prove that there is no k -sum-free subset of \mathbf{N} of upper density on multiples larger than $1/(k+1)$. We use the following theorem of Łuczak and Schoen.

Theorem 3.3 ([LS97]). *Every k -sum-free subset of \mathbf{N} of upper density larger than $1/(k+1)$ is contained in a periodic k -sum-free set.*

Lemma 3.4. *Every k -sum-free $A \subset \mathbf{N}$ satisfies $\tilde{d}(A) \leq 1/(k+1)$.*

Proof. If $\tilde{d}(A) > 0$ then for every N the set $A/N!$ contains a multiple of every natural number. In particular $A/N!$ is not contained in a periodic k -sum-free set, so, by the Łuczak-Schoen theorem,

$$\tilde{d}(A) = \limsup_{N \rightarrow \infty} \bar{d}(A/N!) \leq 1/(k+1). \quad \square$$

4. A FINITARY ŁUCZAK-SCHOEN THEOREM

For a set $A \subset \mathbf{N}$, let $A_x = \{a \in A : a \leq x\}$, let $d_n(A) = |A_n|/n$, and let

$$A_n - (k-1)A_n = \{u - v_1 - \dots - v_{k-1} : u, v_1, \dots, v_{k-1} \in A_n\}.$$

Lemma 4.1. *Suppose that $A \subset \mathbf{N}$ is k -sum-free, that A_{n_0} contains an arithmetic progression $x, x+m, \dots, x+(i-1)m$, and that some $d \in A_{n_0} - (k-1)A_{n_0}$ satisfies $d \equiv x \pmod{m}$. Then for every $\varepsilon > 0$ and every sequence n_1, n_2, \dots of naturals such that $n_{j+1} \geq \varepsilon^{-1}n_j$ for each $j \geq 0$ there is some $\ell \leq kn_0$ such that*

$$d_{n_\ell}(A) \leq \frac{i+k-2}{i(k+1)+k-3} + 4k\varepsilon.$$

Proof. Assume first that $d < x$. Let

$$B = \{a \in A : a + jm \in A \text{ for some } j \in \{1, \dots, i\}\}.$$

Then the sets $A, A + (k-1)x, A \setminus B + (k-1)x + m, \dots, A \setminus B + (k-1)x + (i-1)m$ are pairwise disjoint, so

$$(i+1)|A_n| - (i-1)|B_n| \leq n + (k-1)x + (i-1)m. \quad (*)$$

For each $t \in \mathbf{Z}$ let $u(t)$ be the smallest element of A such that

$$t = u(t) - v_1 - \dots - v_{k-1}$$

for some $v_1, \dots, v_{k-1} \in A$; if no such element exists let $u(t) = \infty$. Since $u(d) \leq n_0$ and $u(x + jm) = \infty$ for each $j \in \{0, \dots, i-1\}$, and since $d \geq -(k-1)n_0$, we may find d' and ℓ such that $1 \leq \ell \leq kn_0$, $u(d') \leq \varepsilon n_\ell$, and $u(d' + jm) > n_\ell$ for each $j \in \{1, \dots, i\}$.

Write $d' = u - v_1 - \dots - v_{k-1}$, where $u, v_1, \dots, v_{k-1} \in A_{\varepsilon n_\ell}$. Then the sets $A, B + v_1 + \dots + v_{k-1}, B + u + v_1 + \dots + v_{k-2}, \dots, B + (k-2)u + v_1, A + (k-1)u$ are pairwise disjoint when restricted to $\{1, \dots, n_\ell - im\}$. Indeed, being k -sum-free certainly A is disjoint from the others, while if for some $0 \leq s < t \leq k-1$ we have

$$(B + su + v_1 + \dots + v_{k-s-1}) \cap (A + tu + v_1 + \dots + v_{k-t-1}) \cap \{1, \dots, n_\ell - im\} \neq \emptyset$$

then there exists $b \in B_{n_\ell - im}$ and $a \in A$ such that

$$b + su + v_1 + \dots + v_{k-s-1} = a + tu + v_1 + \dots + v_{k-t-1},$$

whence

$$d' = (b - a) - (t - s - 1)u - v_1 - \dots - v_{k-t-1} - v_{k-s} - \dots - v_{k-1}.$$

But since $b \in B$ there is some $j \in \{1, \dots, i\}$ such that $b + jm \in A$, and so the above equation shows that $b + jm \geq u(d' + jm) > n_\ell$, contradicting $b \leq n_\ell - im$. Thus

$$2|A_{n_\ell}| + (k-1)|B_{n_\ell}| \leq n_\ell + (k+1)im + k(k-1)u,$$

and the lemma is proved by combining this inequality with (*).

The case $d > x$ is similar, but we consider

$$B = \{a \in A : a - jm \in A \text{ for some } j \in \{1, \dots, i\}\},$$

and we find d' and ℓ such that $1 \leq \ell \leq kn_0$, $u(d') \leq \varepsilon n_\ell$, and $u(d' - jm) > n_\ell$ for each $j \in \{1, \dots, i\}$. \square

Theorem 4.2. *For every $k \geq 2$ and $\varepsilon > 0$ there exist natural numbers $N = N(k, \varepsilon)$ and $Q = Q(k, \varepsilon)$ such that if $n_0 \geq N$ and A is k -sum-free such that*

$$d_{n_0}(A) \geq \frac{1}{k+1} + \varepsilon,$$

then either A_{n_0} is contained in a Q -periodic k -sum-free set, or for every sequence n_1, n_2, \dots of naturals such that $n_{j+1} \geq 16k\varepsilon^{-1}n_j$ for each $j \geq 0$ there is some $\ell \leq kn_0$ such that

$$d_{n_\ell}(A) \leq \frac{1}{k+1} + \varepsilon/2.$$

Proof. Choose $i = i(k, \varepsilon)$ so that

$$\frac{i+k-2}{i(k+1)+k-3} \leq \frac{1}{k+1} + \varepsilon/4,$$

and then choose $N = N(k, \varepsilon)$ and $Q = Q(k, \varepsilon)$ so that if $n_0 \geq N$ every subset of $\{1, \dots, n_0\}$ of size at least $(\varepsilon/2)n_0$ contains an arithmetic progression of length i and common difference dividing Q , and so that

$$\frac{(k-1)k}{k+1} \frac{Q}{N} < \varepsilon/2.$$

The existence of N and Q follows from Szemerédi's theorem.

Suppose $d_{n_0}(A) \geq 1/(k+1) + \varepsilon$. Let

$$R = \{r \in \mathbf{N} : r \equiv a \pmod{Q} \text{ for some } a \in A_{n_0}\},$$

and

$$D = \{r \in R : r + t_1 + \dots + t_{k-1} \notin R \text{ for each } t_1, \dots, t_{k-1} \in R\}.$$

If A_{n_0} is not contained in a Q -periodic k -sum-free set then R must not be k -sum-free. Suppose $x_1, \dots, x_k, x \in R$ and $x_1 + \dots + x_k = x$, where we may assume $x \leq kQ$. Since D is k -sum-free the sets $D, D + x_1 + \dots + x_{k-1}, D + x + x_1 + \dots + x_{k-2}, \dots, D + (k-1)x$ are disjoint, so

$$(k+1)|D_{n_0}| \leq n_0 + (k-1)kQ,$$

whence

$$d_{n_0}(A \setminus D) \geq \varepsilon - \frac{(k-1)k}{k+1} \frac{Q}{n_0} \geq \varepsilon/2.$$

It follows that $A_{n_0} \setminus D_{n_0}$ contains an arithmetic progression $x, x+m, \dots, x+(i-1)m$, where m divides Q . By definition of R and D there exists $u, v_1, \dots, v_{k-1} \in A_{n_0}$ such that $x + v_1 + \dots + v_{k-1} \equiv u \pmod{Q}$. But then $d = u - v_1 - \dots - v_{k-1} \equiv x \pmod{m}$, so by the lemma there is some $\ell \leq kn_0$ such that

$$d_{n_\ell}(A) \leq \frac{i+k-2}{i(k+1)+k-3} + \varepsilon/4 \leq \frac{1}{k+1} + \varepsilon/2. \quad \square$$

We may easily recover the original Łuczak-Schoen theorem from the above finitary version. Indeed, for $k \geq 2$ and $\varepsilon > 0$, let N and Q be as in the above theorem, and suppose A is k -sum-free and $d_{n_i}(A) \geq 1/(k+1) + \varepsilon$ for each i , where $n_i \rightarrow \infty$. By passing to a subsequence of (n_i) we may assume $n_0 \geq N$, $n_{j+1} \geq 16k\varepsilon^{-1}n_j$ for each $j \geq 0$, and, if A is not contained in a Q -periodic k -sum-free set, A_{n_0} is not contained in a Q -periodic k -sum-free set. But then by the above theorem there is some ℓ such that $d_{n_\ell}(A) \leq 1/(k+1) + \varepsilon/2$, a contradiction.

5. FINITARY PROOF OF THEOREM 1

Lemma 5.1. *Fix $\varepsilon > 0$ and let N and Q be as in Theorem 4.2. Then for every $n_0 \geq N$ there exists a finitely supported measure ν on \mathbf{N} such that if A is k -sum-free and A_{n_0} is not contained in a Q -periodic k -sum-free set then $\nu(A) \leq 1/(k+1) + 2\varepsilon$.*

Proof. Continue n_0 to a sequence (n_i) satisfying $n_{i+1} \geq 16k\varepsilon^{-1}n_i$ for each $i \geq 0$, and let (i_s) be a sequence of indices such that $i_{-1} = -1$, $i_0 = 0$, and $i_{s+1} - i_s \geq 2\varepsilon^{-1}kn_{i_s}$ for each $s \geq 0$. Define measures ν_s by

$$\nu_s(A) = \frac{1}{i_s - i_{s-1}} \sum_{i=i_{s-1}+1}^{i_s} d_{n_i}(A),$$

and define ν by

$$\nu(A) = \frac{1}{t+1} \sum_{s=0}^t \nu_s(A),$$

where $t \geq 2\varepsilon^{-1}$.

Suppose that A is k -sum-free, and let s_0 be the least s such that $\nu_s(A) \geq 1/(k+1) + \varepsilon$: if no such s exists then $\nu(A) \leq 1/(k+1) + \varepsilon$, so we are done. Find i_0 such that $i_{s_0-1} < i_0 \leq i_{s_0}$ and $d_{n_{i_0}}(A) \geq 1/(k+1) + \varepsilon$. If $A_{n_{i_0}}$ is not contained in a Q -periodic k -sum-free set then by Theorem 4.2 there are at most kn_{i_0} indices $i > i_{s_0}$ such that $d_{n_i}(A) \geq 1/(k+1) + \varepsilon/2$. Since $kn_{i_0} \leq kn_{i_{s_0}} \leq (\varepsilon/2)(i_{s_0+1} - i_{s_0})$ we find that

$$\nu(A) \leq 1/(k+1) + \varepsilon + 1/(t+1) + \varepsilon/2 \leq 1/(k+1) + 2\varepsilon. \quad \square$$

The next lemma uses an idea based on the contraction mapping theorem which also appeared in [EGM13].

Lemma 5.2. *For every $\varepsilon > 0$ there is a finitely supported measure μ on \mathbf{N} such that every k -sum-free set A satisfies $\mu(A) \leq 1/(k+1) + 4\varepsilon$.*

Proof. Let ν_{n_0} be the measure constructed by the previous lemma for $n_0 \geq N$, and let $\tau : \mathbf{N} \rightarrow \mathbf{N}$ be the map $x \mapsto Qx$. Define measures μ_i inductively as follows.

Start with $\mu_1 = \nu_N$, and thereafter if μ_i is supported on $\{1, \dots, M_i\}$ let

$$\mu_{i+1} = \frac{k}{k+1} \tau_* \mu_i + \frac{1}{k+1} \nu_{QM_i}.$$

If A is k -sum-free and $\nu_{QM_i}(A) > 1/(k+1) + 2\varepsilon$ then by the previous lemma A_{QM_i} is contained in a Q -periodic k -sum-free set. In particular A_{QM_i} is disjoint from $Q\mathbf{N}$, so $\tau_* \mu_i(A) = 0$. Hence in this case $\mu_{i+1}(A) \leq 1/(k+1)$.

If on the other hand $\nu_{QM_i}(A) \leq 1/(k+1) + 2\varepsilon$ then by induction we have

$$\begin{aligned} \mu_{i+1}(A) &\leq \frac{k}{k+1} \left(\frac{1}{k+1} + 2\varepsilon + \left(\frac{k}{k+1} \right)^i \right) + \frac{1}{k+1} \left(\frac{1}{k+1} + 2\varepsilon \right) \\ &= \frac{1}{k+1} + 2\varepsilon + \left(\frac{k}{k+1} \right)^{i+1}. \end{aligned}$$

Hence if i is chosen large enough that $(k/(k+1))^i \leq 2\varepsilon$ then $\mu_i(A) \leq 1/(k+1) + 4\varepsilon$ for every k -sum-free $A \subset \mathbf{N}$. \square

To finish the proof of Theorem 1 we need a version of Lemma 2.1 for measures.

Lemma 5.3. *Suppose that F satisfies*

$$\frac{|(a \cdot F) \triangle F|}{|F|} \leq \varepsilon$$

for every $a \in \{1, \dots, n\}$, and that F has a k -sum-free subset S of size at least $\delta|F|$. Then for every probability measure μ supported on $\{1, \dots, n\}$ there is a k -sum-free set A for which $\mu(A) \geq \delta - \varepsilon$.

Proof. For $x \in F$, let A_x be the set of all $a \in \{1, \dots, n\}$ such that $ax \in S$. Then A_x is k -sum-free, and if we choose $x \in F$ uniformly at random then the expected measure of A_x is

$$\begin{aligned} \mathbf{E}(\mu(A_x)) &= \int \mathbf{P}(a \in A_x) d\mu(a) = \int \mathbf{P}(ax \in S) d\mu(a) \\ &\geq \int \left(\mathbf{P}(x \in S) - \frac{|(a \cdot F) \triangle F|}{|F|} \right) d\mu(a) \\ &\geq \delta - \varepsilon. \end{aligned} \quad \square$$

Theorem 1 follows from the previous two lemmas.

6. FINAL REMARKS

Luczak and Schoen [LS97] also considered so-called strongly k -sum-free sets, sets which are ℓ -sum-free for each $\ell = 2, \dots, k$. They prove that every maximal strongly k -sum-free subset of \mathbf{N} of upper density larger than $1/(2k-1)$ is periodic. Using this theorem one may easily modify Section 3 to verify that for any Følner sequence

(F_m) in (\mathbf{N}, \cdot) the sets F_m contain no strongly k -sum-free subset of size larger than $(1/(2k-1) + o(1))|F_m|$. One may also adapt the methods of Sections 4 and 5 to give an effective proof of this theorem. We leave the details to the energetic reader.

Acknowledgements. I am grateful to Ben Green and to Freddie Manners for helpful comments and discussion.

REFERENCES

- [Bou97] Jean Bourgain. Estimates related to sumfree subsets of sets of integers. *Israel J. Math.*, 97:71–92, 1997.
- [BT13] Vitaly Bergelson and Terence Tao. Multiple recurrence in quasirandom groups. Preprint 2013. Available at <http://arxiv.org/pdf/1211.6372v2.pdf>.
- [EGM13] Sean Eberhard, Ben Green, and Freddie Manners. Sets of integers with no large sum-free subset. Preprint 2013. To appear in *Annals of Mathematics*, available at <http://arxiv.org/pdf/1301.4579v2.pdf>.
- [Erd65] Paul Erdős. Extremal problems in number theory. In *Proc. Sympos. Pure Math., Vol. VIII*, pages 181–189. Amer. Math. Soc., Providence, R.I., 1965.
- [LS97] Tomasz Łuczak and Tomasz Schoen. On infinite sum-free sets of natural numbers. *J. Number Theory*, 66(2):211–224, 1997.

SEAN EBERHARD, CENTRE FOR MATHEMATICAL SCIENCES, UNIVERSITY OF CAMBRIDGE

E-mail address: `s.eberhard@dpms.cam.ac.uk`